Tunneling in the Double Well

Background

As with many of the quantum mechanical systems that have been explored as a part of this workshop, we start here again with the simple harmonic oscillator. This time, however, we have added a second oscillator to the system, so there are effectively two wells. Therefore, the so-called *double-well* potential we are studying here is given by

$$V_{\mathrm{DW}}(x) = \begin{cases} \frac{1}{2}m\omega^2(x+a)^2, & \text{if } \infty \leq x \leq 0 \\ \frac{1}{2}m\omega^2(x-a)^2, & \text{if } 0 \leq x \leq \infty. \end{cases} \tag{1}$$

As we shall see, the behavior of this system is quite interesting. By studying this system, one can explore how superposition states can lead to tunneling behavior, and we can observe the timescale of this behavior. We will also explore the phenomenon of *resonant tunneling* of *localized* states.

What is a localized state? For this exercise, we define a localized state to be a state that is contained (almost) entirely within one well. That is, a state localized to the left well will have a negligibly low probability density in the right well, and vice versa.

The flowfile

The flowscene has scopes that calculate the potential and resulting eigensystem given parameters that you can input. The harmonic frequency is fixed (hidden within a scope) at $\omega = 10$, but on the left side of the scene you can access parameters governing the spacing between the two wells and the relative tilt of the wells. Above the *Time Evolution* scope there are *Scalar outputs* that allow you to see the energy of the first three eigenstates of the potential as well as their energy differences.

Below the scope that builds the potential, you can see a *Linear Combination* block that you can use to build the initial state that you will use in the *Time Evolution* scope. Here, each row corresponds to an eigenstate, and each column corresponds to a real (left column) or imaginary (right column) number that you can enter. These values represent the (un-normalized) real and imaginary coefficients for each eigenstate that makes up the resulting linear combination.

For example, given the first three eigenstates of the potential $\psi_i(x)$, $i = 0, .., 2$, if you wanted to create a state that looked like

$$\psi(x) = 1\frac{1}{\sqrt{2}}(\psi_0(x) - i\psi_2(x)) \tag{2}$$

you would fill out the *Linear Combination* with a 1 in the top left box (ground state, real part), a -1 in the bottom right box (second excited state, imaginary part). The other boxes would be zero. By checking the *Normalize output* box, you can ensure that your state will always be normalized, so you do not have to enter the factor of $1/\sqrt{2}$, just the relative populations in each state.

1 The two-well system

First, we will explore the un-tilted two-well system. The spacing between wells should be set to 0.75.

1. By changing the numbers in the *Linear Combination* box, look at the ground and first excited states. What are the energies of these states? How do the energies of the states change as you change the separation between wells? Why is this happening? Given that in Composer, $\hbar = 1$ and $\omega = 10$, do the values of E_0 and E_1 make sense?

2. Now return the spacing between wells to the initial value of 0.75. Based on what you observe, what are the two linear combinations that would give you states localized in the left and right well?

3. Initialize a state localized in the left well, then click the *Play* button. What happens? Explain why the motion is periodic.

4. Now change the spacing between wells. How do the dynamics change? How does the timescale of the dynamics change as a function of the well spacing? Be as quantitative as possible here.

2 The tilted two-well system

Now we can explore what happens as we add a tilt to one well relative to the other.

1. Change the value of the well tilt parameter. What happens to the energy spacing between the ground and first excited state, $E_1 - E_0$? What about $E_2 - E_1$? What happens to the ground state energy when the well tilt is larger than E_0? Why does E_0 converge to the value it does?

2. For a well tilt of 5, and considering only the excited states ψ_1 and ψ_2, what do the localized states look like? What are the linear combinations you need to get states localized in the left well? The right well?

3. Keeping the well tilt the same, if you start with a state localized in the right well, do you see tunneling? Why or why not?

4. Why is the situation different for a well tilt of 10? Explain quantitatively why this well tilt gives rise to interesting behavior. What is happening to E_1 and E_2 as you increase the well tilt to this value?

5. For a well tilt of 10, how can you create states from ψ_1 and ψ_2 that are localized in each well? Do these states exhibit tunneling? Why or why not?

6. How does the tunneling time compare to the tunneling time in Problem 1? Is it shorter or longer, for a given well separation? Is the scaling with the well separation the same as in Problem 1?

7. What changes if you change the well tilt to -10?

8. For what other values of the well tilt do you expect to see resonant tunneling behavior? Why? How does the system behavior change as you move to higher and higher well tilts?